

# SYNTOMIC REGULATORS AND $p$ -ADIC INTEGRATION II: $K_2$ OF CURVES

BY

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## ABSTRACT

We show that the ad hoc regulator of Coleman and de Shalit for elements of  $K_2$  of curves evaluated on a holomorphic differential is the same as the syntomic regulator of the same elements cup produced with the same differential. Combined with the results of Coleman and de Shalit this gives a relation between syntomic regulators and special values of  $p$ -adic  $L$ -functions. The main technical innovation is the notion of a local index — a kind of generalized residue.

## 1. Introduction

Let  $C/\mathbb{C}$  be a smooth complete curve,  $f, g \in \mathbb{C}(C)$  two rational functions on  $C$ . In [Bei80], Beilinson defines the complex regulator of  $f$  and  $g$ ,

$$r_C(\{f, g\}) \in H^1(C^{an}, \mathbb{R}(1)),$$

such that the following formula is satisfied when  $\omega \in H^0(C, \Omega_{C/\mathbb{C}}^1)$  is a holomorphic 1-form on  $C$ :

$$(1.1) \quad r_C(\{f, g\}) \cup [\omega] = \frac{1}{2\pi i} \int_{C(\mathbb{C})} \log |g|^2 \overline{\mathrm{d}\log f} \wedge \omega.$$

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One can show that  $r_C$  is antisymmetric and satisfies the Steinberg relation and therefore defines a map

$$r_C: K_2(\mathbb{C}(C)) \rightarrow H^1(C^{an}, \mathbb{R}(1)).$$

Beilinson, following Bloch, shows the following theorem:

**THEOREM 1** (Bloch, Beilinson): *If  $E/\mathbb{Q}$  is an elliptic curve with complex multiplication, then there exist  $f, g \in \mathbb{Q}(E)$  such that*

$$(1.2) \quad r_E(\{f, g\}) \cup [\omega] = a_{f,g} \Omega L^*(E, 0),$$

where  $\omega \in H^0(E, \Omega_{E/\mathbb{Q}}^1)$ ,  $a_{f,g} \in \mathbb{Q}$  and  $\Omega$  is a known transcendental period. The  $L$ -function  $L^*(E, s)$  is the usual  $L$ -function multiplied by the Gamma factor.

In [CdS88], Coleman and de Shalit gave a  $p$ -adic analogue of formula (1.1). Suppose that  $C/\mathbb{C}_p$  is a smooth complete curve with good reduction (the results of [CdS88] apply in some greater generality). Then they define a  $p$ -adic regulator

$$r_{p,C}: K_2(\mathbb{C}_p(C)) \rightarrow \text{Hom}(H^0(C, \Omega_{C/\mathbb{C}_p}^1), \mathbb{C}_p).$$

The value of  $r_{p,C}$  on a symbol  $\{f, g\}$  with  $f, g \in \mathbb{C}_p(C)$  is defined to be the functional on holomorphic 1 forms given by

$$(1.3) \quad r_{p,C}(\{f, g\})(\omega) := \int_{(f)} \log(g) \cdot \omega.$$

Briefly, the meaning of this formula is as follows: Coleman’s  $p$ -adic integration theory allows (as will be explained below) to define a function

$$F_{\log(g) \cdot \omega} = \int \log(g) \cdot \omega: C(\mathbb{C}_p) \rightarrow \mathbb{C}_p,$$

unique up to a constant. If the divisor of  $f$  is  $(f) = \sum n_i(x_i)$ , then one defines

$$\int_{(f)} \log(g) \cdot \omega := \sum n_i F_{\log(g) \cdot \omega}(x_i).$$

Coleman and de Shalit used their regulator to derive an analogue of Theorem 1.

**THEOREM 2** (Coleman, de Shalit): *For the same  $E, f, g,$  and  $\omega$  as in Theorem 1, let  $p$  be a prime that splits in the CM field of  $E$ . Then*

$$r_{p,E}(\{f, g\})(\omega) = a_{f,g} \Omega_p L_p(E, 0),$$

where  $L_p(E, s)$  is the  $p$ -adic  $L$ -function of  $E$ ,  $a_{f,g}$  is the same as in Theorem 1 and  $\Omega_p$  is a  $p$ -adic period (into which we have pushed an Euler factor at  $p$  to keep the exposition simple).

We remark that the point 0 is outside the interpolation range, so the  $p$ -adic formula cannot be recovered from the complex one.

The formula (1.3) is derived in an ad hoc manner from its complex counterpart (1.1). However, the relation with the value of a  $p$ -adic  $L$ -function seems to indicate that it is in some sense the “correct”  $p$ -adic formula. On the other hand, there is a general method of assigning  $p$ -adic regulators to elements of  $K$ -theory, namely the construction of syntomic regulators. The purpose of this work is to show that on  $K_2$  of curves these constructions are very closely related.

Let  $L/\mathbb{Q}_p$  be a finite extension with residue field  $\kappa$  and let  $Z/\mathcal{O}_L$  be smooth and surjective (i.e., not an  $L$  scheme) with generic fiber  $Z_L$  and special fiber  $Z_\kappa$ . Then Gros [Gro90] and Nizioł [Niz97] define regulators (= Chern characters) from the  $K$ -theory of  $Z$  into syntomic cohomology. This last cohomology has several versions by now. They are all essentially the same in the projective case. We will use the version developed by us in [Bes98a]. With the notation of loc. cit., the regulator takes the form

$$ch_{i,j}: K_j(Z) \rightarrow H_{\text{syn}}^{2i-j}(Z, i).$$

In this work we will only be interested in the case where  $i = j = 2$  and  $Z$  is projective and of relative dimension 1. Then it can be shown that

$$H_{\text{syn}}^2(Z, 2) \cong H_{\text{dR}}^1(Z_L/L).$$

As  $Z_\kappa$  is a smooth and projective curve over a finite field, it follows from [Har77] that  $K_i(Z_\kappa)$  is torsion for  $i \geq 1$ . The localization sequence in  $K$ -theory shows that

$$K_2(Z) \otimes \mathbb{Q} \cong K_2(Z_L) \otimes \mathbb{Q}.$$

Since the target of the Chern character is an  $L$ -vector space, it induces a map

$$r_{\text{syn}}: K_2(Z_L) \otimes \mathbb{Q} \rightarrow H_{\text{dR}}^1(Z_L/L).$$

Let  $C = Z_L \otimes \mathbb{C}_p$ . Then one can write the following diagram:

$$\begin{array}{ccc}
 (1.4) & K_2(Z_L) \otimes \mathbb{Q} & \xrightarrow{r_{\text{syn}}} & H_{\text{dR}}^1(Z_L/L) \\
 & \downarrow & & \downarrow \\
 & K_2(C) \otimes \mathbb{Q} & & H_{\text{dR}}^1(C/\mathbb{C}_p) \\
 & \downarrow & & \downarrow \text{Poincaré duality} \\
 & K_2(\mathbb{C}_p(C)) \otimes \mathbb{Q} & & \text{Hom}(H_{\text{dR}}^1(C/\mathbb{C}_p), \mathbb{C}_p) \\
 & \downarrow r_{p,C} & & \downarrow \\
 & \text{Hom}(H^0(C, \Omega_{C/\mathbb{C}_p}^1), \mathbb{C}_p) & \equiv & \text{Hom}(H^0(C, \Omega_{C/\mathbb{C}_p}^1), \mathbb{C}_p).
 \end{array}$$

Here, the map given by Poincaré duality is (to fix signs)

$$\alpha \mapsto (\beta \mapsto \text{tr}(\beta \cup \alpha)).$$

Our main result is then

**THEOREM 3:** *The diagram above commutes.*

Note that what the theorem says is essentially that the regulator of Coleman and de Shalit computes “part” of the syntomic regulator, namely the value of the syntomic regulator, thought of as a functional on  $H_{\text{dR}}^1$  via Poincaré duality, on the subspace of holomorphic forms. The theorem of Coleman and de Shalit suggests that, rather than that their regulator gives only part of the information, it is in fact the syntomic regulator that should be modified to land in a smaller subspace. In the corresponding complex situation, one uses the action of complex conjugation to cut down the target space to the right size. Is there a similar procedure in the  $p$ -adic situation?

The proof of the main theorem turns out to be fairly simple, once one makes explicit the two sides of diagram (1.4). The new idea involved is a kind of a residue theorem, Corollary 4.11, which is similar in spirit to the reciprocity law proved by Coleman in [Col89].

The structure of the paper is as follows: In section 2 we recall the basics of the theory of Coleman integration we need from [CdS88] and in particular explain the formula (1.3) for the  $p$ -adic regulator. In section 3 we use the theory developed in [Bes98a] to write an explicit one-form representing the syntomic regulator of an element of  $K_2$  on an open part of  $Z$ . Using the formula of Serre for the cup product in de Rham cohomology on a curve we reduce the proof of the main

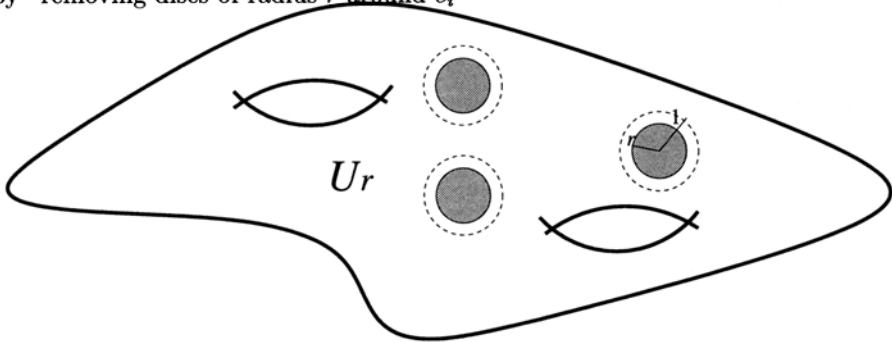
theorem to a formula (Proposition 3.4) relating residues and Coleman integrals. In section 4 we define **local indices**, which are some kind of a generalization of residues that make sense out of the residue of the log function in some cases. Our main reciprocity law, Proposition 4.10, should be considered as an extension of the residue theorem to these generalized residues. The final section verifies Proposition 3.4 and thereby completes the proof of the main theorem.

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**2. *p*-adic integration**

Our basic setup throughout this work will be as follows:  $K$  is a complete subfield of  $\mathbb{C}_p$  with ring of integers  $\mathcal{O}_K$  and residue field  $\kappa$ . Let  $X/\mathcal{O}_K$  be a smooth projective and surjective scheme of relative dimension 1 with generic fiber  $X_K$  and special fiber  $X_\kappa$ . Let  $Y \subset X$  be an open affine subscheme, smooth and surjective over  $\mathcal{O}_K$ . The special fiber of the complement of  $Y$  is a union of a finite number of points:  $X_\kappa - Y_\kappa = \{e_1, \dots, e_n\}$ .

To the situation above one associates a “basic wide open”, in Coleman’s terminology (see [CdS88, 2.1]). For  $r < 1$ , consider the rigid space  $U_r$  obtained by “removing discs of radius  $r$  around  $e_i$ ”



The formal definition of  $U_r$  is, following Berthelot, as follows: If  $Y_\kappa$  is locally in  $X_\kappa$  given by the equation  $\bar{h} = 0$ , with  $\bar{h}$  the reduction of a function  $h$  on  $X$ , then  $U_r$  is locally given by the inequality  $|h| > r$ . While the definition depends on the choice of the local lifts  $h$ , the inverse limit  $U = \varprojlim_{r \rightarrow 1} U_r$  does not,

in the sense that any two choices of local lifts give the same  $U_r$  for sufficiently large  $r$  [Ber96]. The spaces of functions and one-forms,  $A(U) := \varinjlim_{r \rightarrow 1} A(U_r)$  and  $\Omega^1(U) := \varinjlim_{r \rightarrow 1} \Omega^1(U_r)$ , are the spaces of “overconvergent” functions and one-forms. Inside  $U$  one has the “underlying affinoid” in Coleman’s terminology, which is the space obtained by throwing away from  $X_K$  the corresponding open discs of radius 1.

We will need the notion of Frobenius endomorphisms on  $U$ . One first defines these on  $Y_\kappa$ . We have some  $Y'/F_q$ , for some  $q = p^r$  such that  $Y_\kappa \cong Y' \otimes_{F_q} \kappa$ . Consider the  $\kappa$ -morphism  $\varphi = \text{Fr}_{Y'}^r \otimes \text{id}_\kappa$ .

*Definition 2.1:* A **Frobenius endomorphism** of  $Y_\kappa$  is any  $\varphi$  obtained in the way described above. The integer  $q$  will be called the degree of the Frobenius endomorphism.

An important remark is that any power of a Frobenius endomorphism is again a Frobenius endomorphism.

**THEOREM 2.2** (Coleman [CdS88, Theorem 2.2.]): *For any Frobenius endomorphism  $\varphi$  there exists a rigid analytic map  $\phi: U \rightarrow U$  reducing to  $\varphi$  (i.e.,  $\phi: U_r \rightarrow U_s$  with  $s < r$  sufficiently near 1). Any such  $\phi$  will be called a Frobenius endomorphism of  $U$ .*

The proof in loc. cit. is for  $K = \mathbb{C}_p$  but works in general. We will call a map  $\phi$  as above a Frobenius endomorphism of  $U$ . Clearly one obtains an operator  $\phi^*$  on  $A(U)$  and  $\Omega^1(U)$ .

*Example 2.3:* For  $X = \mathbb{P}_{\mathcal{O}_{\mathbb{C}_p}}^1$  and  $Y = \mathbb{G}_m$  we can take  $U_r = \{r < |z| < 1/r\}$ ,  $\phi(z) = z^p$ ,  $\phi: U_r \rightarrow U_{r^p}$ .

We now sketch Coleman’s integration theory on  $U$ . See [CdS88] for a full account. For simplicity and compatibility with loc. cit. we will assume from now until the end of this section that  $K = \mathbb{C}_p$ , hence  $\kappa = \bar{F}_p$ , and set  $C = X_K$ . The space  $U$  decomposes **set theoretically** into a disjoint union of residue discs  $U_x$  over  $x \in X(\bar{F}_p)$ . When  $x \in Y(\bar{F}_p)$ ,  $U_x$  is the collection of closed points of  $C$  reducing to  $x$  and is isomorphic to the open unit disc  $\{|z| < 1\}$  (because  $X$  is smooth) via some local parameter which we denote  $z_x$ . For each  $r < 1$  and each  $e_i$ , the residue disc of  $e_i$  in  $U_r$  is the collection of closed points in  $U_r$  reducing to  $e_i$  and is isomorphic to an open annulus  $\{r < |z| < 1\}$  via a local parameter  $z_{e_i}$ . Here we assume that  $z_{e_i}$  extends to a local parameter for the residue disc of  $e_i$  in  $C$ . This fixes an **orientation** for the annulus  $e_i$  (see [Col89, II] and the discussion below in section 4). The pro-space  $U_{e_i}$  is the inverse limit of these

annuli. We have

$$\Omega^1(U_x) = A(U_x)dz_x$$

and

$$A(U_x) = \{f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ converging for } |z| < 1\},$$

when  $x \in Y_s(\bar{F}_p)$ , or

$$A(U_x) = \{f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \text{ converging for } r < |z| < 1 \text{ for some } r < 1\},$$

when  $x = e_i$  for some  $i$ . In both cases we have set  $z = z_x$ .

*Definition 2.4:* The annuli  $U_{e_i}$ , with the orientation discussed above, are called the annuli ends of  $U$  and their collection is denoted  $End(U)$ .

In fact, we will abuse the notation and will usually write  $e_i$  for  $U_{e_i}$ . To confuse things further, we will sometime use  $e_i$  to refer to the full residue disc of  $e_i$  in  $C$ . The intention should be clear from the context.

*Definition 2.5:* The residue of the form  $\omega = \sum_{n=-\infty}^{\infty} a_n z_{e_i}^n dz_{e_i}$  along the annuli  $e_i$  is given by

$$Res_{e_i} \omega = a_{-1}.$$

The residue is independent of the choice of parameter.

Coleman’s theory allows one to integrate certain locally analytic one-forms. One first needs to make a choice of a branch of the  $p$ -adic logarithm.

*Definition 2.6:* A branch of the  $p$ -adic logarithm is any locally analytic homomorphism  $\log : \mathbb{C}_p^\times \rightarrow \mathbb{C}_p$  with the usual expansion for  $\log$  around 1. Such a function is determined by choosing  $\pi \in \mathbb{C}_p$  such that  $|\pi| < 1$  and declaring  $\log(\pi) = 0$ .

Suppose a branch of the  $p$ -adic logarithm has been chosen. We define  $A_{log}(U_x)$  to be  $A(U_x)$  if  $x \in Y_s(\bar{F}_p)$  and to be the polynomial ring in the function  $\log(z_{e_i})$  over  $A(U_{e_i})$  if  $x = e_i$ . This ring is independent of the choice of  $z_{e_i}$  because it can be shown that the difference of the logs of two local parameters is analytic on  $U_{e_i}$ . Set  $\Omega_{log}^1(U_x) := A_{log}(U_x)dz_x$ . Then one defines locally analytic functions and one-forms on  $U$  by

$$A_{loc}(U) := \prod_x A_{log}(U_x), \quad \Omega_{loc}^1(U) := \prod_x \Omega_{log}^1(U_x).$$

There is an obvious differential  $d: A_{loc}(U) \rightarrow \Omega^1_{loc}(U)$ . One easily checks that this is surjective. The point is that by adding logs we are able to integrate  $dz/z$ . The inverse image under  $d$  is very big, because one can choose a different constant of integration at every  $x$ . Coleman’s theory isolates a subclass of locally analytic differential forms that can be integrated uniquely up to a global constant. This is done using the Frobenius endomorphism  $\phi$ . It acts on locally analytic functions and differential forms in a way compatible with the differential  $d$ . Coleman’s idea is now as follows: One constructs a certain subspace  $M(U)$  of  $A_{loc}(U)$ , which we call the space of **Coleman functions**, and a vector space map (integration), which we denote by  $\int$  or by  $\omega \mapsto F_\omega$ , from  $M(U) \otimes_{A(U)} \Omega^1(U)$  to  $M(U)/\mathbb{C}_p \cdot 1$ . The map  $\int$  is characterized by three properties:

1. It is a primitive for the differential in the sense that  $dF_\omega = \omega$ .
2. It is Frobenius equivariant in the sense that  $\int(\phi^*\omega) = \phi^* \int(\omega)$ .
3. If  $g \in A(U)$ , then  $F_{dg} = g + \mathbb{C}_p$ .

The construction relies on a simple principle: If  $\int$  has already been defined on some space  $W$ , and  $\omega \in \Omega^1_{loc}(U)$  is such that there is a polynomial  $P(t)$  with  $\mathbb{C}_p$  coefficients such that  $P(\phi^*)\omega = \eta \in W$ , then the conditions on the integral force the equality  $P(\phi^*)F_\omega = F_\eta + \text{Const}$ . When  $P$  has no roots of unity as roots this condition fixes  $F_\omega$  up to a constant. Starting with  $W_0 = dA(U)$  one finds a unique way of integrating all  $\omega \in W_1 = \Omega^1(U)$ . One defines recursively  $W_{i+1} = (\int(W_i)) \cdot \Omega^1(U)$  and checks that the principle above permits extending  $\int$  uniquely to  $W_{i+1}$ . Finally one sets  $M(U) = \bigcup_i \int W_i$ . The entire theory turns out to be independent of the choice of  $\phi$ .

In particular, suppose  $\omega, \eta$  are rigid forms on  $U$ . Coleman’s theory then finds a canonical (up to constant)  $F_\eta: U \rightarrow \mathbb{C}_p$  such that  $dF_\eta = \eta$ , and  $F_{F_\eta \cdot \omega}$  such that  $dF_{F_\eta \cdot \omega} = F_\eta \cdot \omega$ . One can show that if  $g \in A(U)$ , then  $F_{d\log(g)} = \log(g)$ . Now we can continue to define

$$F_{\log(g) \cdot \omega} = F_{F_{d\log(g)} \cdot \omega}.$$

Coleman and de Shalit show that for a rational function  $g \in \mathbb{C}_p(C)$  there is a canonical extension of  $F_{\log(g) \cdot \omega}$  to  $C - \{x: \text{ord}_x g \neq 0\}$ . The extension is obtained by covering this set by basic wide opens and gluing the resulting integrals.

In some situations, notably when  $f$  and  $g$  in (1.3) have a common singular point, we will need to extend  $F_{\log(g) \cdot \omega}$  to a singular point  $x_0$  of  $g$  as well. The extension is not part of a general theory but is done ad hoc in [CdS88, 3.2]. As we will see in Proposition 5.5, it is in fact the correct choice.

*Definition 2.7:* Suppose  $\text{ord}_{x_0}(g) \neq 0$  and a choice of  $F_{\log(g) \cdot \omega}$  has been made. We define  $F_{\log(g) \cdot \omega}(x_0)$  as follows: We choose an integral  $F_\omega$  such that  $F_\omega(x_0) = 0$ .



Then we choose  $\int F_\omega \operatorname{dlog} g$  in such a way that the integration by parts formula is satisfied, i.e.,

$$F_{\log(g)\cdot\omega} + \int F_\omega \operatorname{dlog} g = \log(g) \cdot F_\omega.$$

We now define  $F_{\log(g)\cdot\omega}(x_0) = -(\int F_\omega \operatorname{dlog} g)(x_0)$ .

The motivation for this formula is that we should expect  $F_\omega \log g$  to tend to 0 as  $x$  tends to  $x_0$  just as in the complex case. If everything is defined over a discrete valuation field, then Coleman and de Shalit show that Definition 2.7 gives the unique continuous extension. Another motivation is given by the following easy lemma

LEMMA 2.8: *With the extension of Definition 2.7 the function sending  $(g, \omega)$  to  $F_{\log(g)\cdot\omega}(x_0)$  is bilinear in the following sense: if we are given forms  $\omega$  and  $\eta$  and we choose  $F_{\log(g)\cdot\omega}$ ,  $F_{\log(g)\cdot\eta}$  and  $F_{\log(g)\cdot(\omega+\eta)} = F_{\log(g)\cdot\omega} + F_{\log(g)\cdot\eta}$ , then*

$$F_{\log(g)\cdot\omega}(x_0) + F_{\log(g)\cdot\eta}(x_0) = F_{\log(g)\cdot(\omega+\eta)}(x_0).$$

*A similar formula holds if we fix  $\omega$  and consider functions  $g_1, g_2$  and  $g_3 = g_1 \cdot g_2$ . In addition, the extension coincides with the integral if  $\operatorname{ord}_x(g) = 0$ .*

The formula (1.3) for the  $p$ -adic regulator is now clear.

### 3. The syntomic regulator

In this section we compute an explicit representative for the syntomic regulator evaluated on an element of  $K_2$  of a curve. We begin by reviewing some of the results of [Bes99] in the present context. Rather than discussing syntomic cohomology here, we will work with the **modified syntomic cohomology** introduced in [Bes99, Section 8].

We keep the notation introduced in the previous section but assume for the moment that the field  $K$  is a finite extension of  $\mathbb{Q}_p$ . Recall that  $X$  is a smooth projective  $\mathcal{O}_K$ -scheme,  $Y$  is an open subscheme of  $X$ , smooth and surjective over  $\mathcal{O}_K$ , and that to  $Y$  one associates a basic wide open  $U$ . We fix a Frobenius endomorphism  $\phi: U \rightarrow U$ . According to [Bes99, 8.6.3] there is a canonical isomorphism,  $H_{\text{syn}}^2(X, 2) \xrightarrow{\sim} H_{\text{ms}}^2(X, 2)$ , which is compatible with Chern classes by definition. Since we will only work with this degree of cohomology there is no need to introduce syntomic cohomology and we may work with the modified variant throughout. As  $Y$  is affine, we can write part of its modified syntomic

cohomology quite easily. Namely, by [Bes99, 10.1.1] we have

$$H_{\text{ms}}^i(Y, i) = \varinjlim_k \left\{ \omega \in \Omega^i(Y_K/K)_{\log}, h \in \Omega^{i-1}(U)/d\Omega^{i-2}(U): dh = \left( 1 - \left( \frac{\phi^*}{q^i} \right)^k \right) \omega \right\}.$$

The notation differs somewhat from loc. cit.: what we denote here by  $\Omega^j(U)$  is denoted by  $\Omega_{A^+, K}^j$  there, where  $A$  is a weak complete closure of  $Y$  in the sense of Monsky and Washnitzer. Also,  $\Omega^i(Y_K/K)_{\log}$  is the space of degree  $i$  algebraic differential forms on  $Y_K$  with logarithmic singularities along  $X_K - Y_K$ . We have also identified  $\omega \in \Omega^i(Y_K/K)_{\log}$  with its pullback under  $U \hookrightarrow Y_K$ . The connecting map from the  $k$ -th level to the  $km$ -th level is given by

$$(\omega, h) \mapsto \left( \omega, \sum_{s=0}^{m-1} (\phi^*/q^n)^{sk} h \right).$$

One can show directly that this definition is independent up to isomorphism of all choices. In particular, it is clear that it is unchanged if we replace  $\phi$  by some power  $\phi^k$ .

We next recall the computation of the modified syntomic regulator for functions, i.e.,  $ch_{1,1}: \mathcal{O}_Y^\times \rightarrow H_{\text{ms}}^1(Y, 1)$ . We have  $ch_{1,1} = -c_1^1$  so it is enough to compute the Chern class  $c_1^1$ . Suppose we are given  $f \in \mathcal{O}_Y^\times$ . By replacing  $\phi$  by  $\phi^k$  for some  $k$  if necessary we can assume  $\bar{f} \circ \bar{\phi} = \bar{f}^q$ . We will find  $c_1^1(f)$  in the first term of the directed system, which is given explicitly by

$$\{(\omega, h): \omega \in \Omega^1(Y_K)_{\log}, h \in A(U), dh = (1 - \phi^*/q)\omega\}.$$

Set  $f_0 := f^q/\phi^* f$ . Then  $f_0 \equiv 1$  modulo the maximal ideal of  $\mathcal{O}_K$  and therefore  $\log f_0 \in A(U)$  is a well defined rigid function. According to [Bes99, Prop. 10.3]

$$ch_1^1(f) := (d\log f, \frac{1}{q} \log f_0).$$

To define regulators on symbols we need to recall the cup product in modified syntomic cohomology. We again only say what it is in the case we need, i.e.,  $H_{\text{ms}}^1(Y, 1) \times H_{\text{ms}}^1(Y, 1) \rightarrow H_{\text{ms}}^2(Y, 2)$ . In this case, according to [Bes99, 10.1.2], it is given on representatives as above by the formula

$$(\omega_1, h_1) \cup (\omega_2, h_2) = (\omega_3, h_3)$$

with

$$\omega_3 = \omega_1 \wedge \omega_2, \quad h_3 = (1/q)h_1\phi^*\omega_2 - h_2\omega_1.$$

The relevant condition, which the reader may easily check for himself, for example using the trick below, is  $dh_3 = (1 - \phi^*/q^2)\omega_3$ . This is only one of a family of possible homotopic cup products but we make no use of any other possibility.

*Remark 3.1:* The following trick is quite helpful in computations: Consider a field  $F$  and two vector spaces  $V$  and  $W$  over  $F$ , and let  $T$  and  $S$  be operators on  $V$  and  $W$  respectively. The polynomial ring  $F[t, s]$  acts on  $V \otimes W$  by letting  $t$  act as  $T \otimes 1$  and  $s$  as  $1 \otimes S$ . The operator  $T \otimes S$  then corresponds to the action of  $ts$ . We will use this in a situation where  $V$  and  $W$  are spaces of differential forms (or functions) and both  $S$  and  $T$  correspond to the action of Frobenius. Then  $V \otimes W$  maps to a space of differential forms and  $T \otimes S$  corresponds to the action of Frobenius on this space. This allows one to translate relations in  $F[t, s]$  to relations on the Frobenius action. The main example to be used in this paper is the relation

$$1 - \frac{ts}{q^2} = \left(1 - \frac{t}{q}\right) \frac{s}{q} + \left(1 - \frac{s}{q}\right),$$

which implies that for a function  $f$  and a one-form  $\omega$  we have

$$(3.1) \quad \left(1 - \frac{\phi^*}{q^2}\right)(f\omega) = \left(1 - \frac{\phi^*}{q}\right)f \cdot \frac{\phi^*}{q}\omega + f \cdot \left(1 - \frac{\phi^*}{q}\right)\omega.$$

All the above was true for  $Y$  of arbitrary dimension. In our case, taking into account that  $Y$  is of relative dimension 1, we obtain the following formula for the Chern character evaluated on the symbol  $\{f, g\}$  with  $f, g \in \mathcal{O}_Y^\times$ : As the Chern character is multiplicative we have

$$ch_{2,2}(\{f, g\}) = (d\log f, (1/q) \log f_0) \cup (d\log g, (1/q) \log g_0) = (0, \eta_0(f, g)),$$

with

$$(3.2) \quad \eta_0(f, g) = \frac{1}{q^2} \log f_0 \, d\log \phi^* g - \frac{1}{q} \log g_0 \, d\log f.$$

From the description of the modified syntomic cohomology above it is evident that  $H_{ms}^2(Y, 2) \cong H^1(U)$ . The naive isomorphism has to be twisted, however, to get the correct identification, a point which becomes clear once one considers the way the naive identification changes with  $k$ . The following result is a special case of [Bes99, 8.6.3 and 10.1.3] and is given here only to fix notation.

**PROPOSITION 3.2:** *There exists a natural isomorphism  $H^1(U) \rightarrow H_{ms}^2(Y, 2)$ . On the  $k$ -th level of the direct limit this isomorphism is given by the map*

$$\eta \mapsto (0, (1 - (\phi^*/q^2)^k)\eta).$$

*Proof:* It is clear that the map above commutes with all the transition maps. It is an isomorphism because the eigenvalues of  $\phi^*$  on  $H^1(U)$  are Weil numbers with absolute values  $q^{1/2}$  or  $q$  by [CdS88, 2.5]. ■

For easy reference we write down the resulting formula for the image of the regulator on the symbol  $\{f, g\}$  in de Rham cohomology.

**PROPOSITION 3.3:** *Let  $f, g \in \mathcal{O}_Y^\times$  and suppose we have chosen a Frobenius endomorphism  $\phi$  of degree  $q$  such that  $\bar{f} \circ \bar{\phi} = \bar{f}^q$  and  $\bar{g} \circ \bar{\phi} = \bar{g}^q$ . Then the image of  $ch_{2,2}(\{f, g\})$  in  $H^1(U)$  is given by the class of any form  $\eta(f, g) \in \Omega^1(U)$  satisfying*

$$(3.3) \quad (1 - \phi^*/q^2)\eta(f, g) = \eta_0(f, g) + d(\ ).$$

Suppose now that  $K = \mathbb{C}_p$ . While we expect the theory of modified syntomic cohomology to work in this case as well, a verification of this possibility requires some facts in rigid cohomology which are not known to us. Nevertheless, the explicit computations we have performed in this section do not depend on the assumption that  $K$  is finite over  $\mathbb{Q}_p$  and we may simply take Proposition 3.3 as a definition of a modified syntomic regulator for symbols on  $Y$ . In particular, in the formulation of the following key result we are free to assume that  $f$  and  $g$  are defined over  $\mathbb{C}_p$ .

We now explain the main result leading to the proof of Theorem 3. Suppose again that  $K = \mathbb{C}_p$  and denote, as in section 2, the generic fiber of  $X$  by  $C$ . For  $f, g \in \mathbb{C}_p(C)$  and  $x \in C$  let  $t_x(f, g)$  be the tame symbol of  $f$  and  $g$  at  $x$ , defined by

$$t_x(f, g) = \left[ (-1)^{\text{ord}_x(f) \text{ord}_x(g)} f^{\text{ord}_x(g)} g^{-\text{ord}_x(f)} \right] (x).$$

The tame symbol is known to satisfy the Steinberg relations and therefore to extend to a map  $t_x: K_2(\mathbb{C}_p(C)) \rightarrow \mathbb{C}_p^\times$ . It is well known that an element in  $K_2(\mathbb{C}_p(C))$  extends to  $K_2(C)$  if and only if its tame symbols at all points of  $C$  are 1.

**PROPOSITION 3.4:** *Let  $f, g \in \mathbb{C}_p(C)$  and let  $\omega \in H^0(C, \Omega^1_{C/\mathbb{C}_p})$ . Then*

$$(3.4) \quad \sum_{e \in \text{End}(U)} \text{Res}_e(F_\omega \cdot \eta(f, g)) = \sum_{x \in C} (\log t_x(f, g)) \cdot F_\omega(x) + \int_{(f)} \log g \cdot \omega.$$

The next two sections will be devoted to the proof of this proposition.

We may now turn to the proof of Theorem 3. We suppose again that  $K$  is finite over  $\mathbb{Q}_p$ . We first prove a lemma.

LEMMA 3.5: *Let  $Y$  be a smooth scheme, surjective and of relative dimension 1 over  $\mathcal{O}_K$ , and let  $Y'$  be a closed proper subscheme. Then the composition*

$$(3.5) \quad K'_2(Y') \rightarrow K_2(Y) \xrightarrow{ch_{2,2}} H^2_{ms}(Y, 2)$$

is 0.

*Proof:* If  $Y'$  is supported in the closed fiber, then the result follows easily because  $K_2(Y'_\kappa)$  is torsion by [Har77]. If it is not, then, using the compatibility of the modified syntomic regulator with base change [Bes99, 8.8], we are reduced to the case where  $Y'$  is the image of a section  $\text{Spec } \mathcal{O}_K \rightarrow Y$ . In particular,  $Y'$  is smooth over  $\mathcal{O}_K$ . It is well known that under this assumption we may factor the pushforward map in  $K$ -theory,  $K'_2(Y') \rightarrow K_2(Y)$ , as

$$K'_2(Y') \cong K_2(Y, Y - Y') \rightarrow K_2(Y),$$

where  $K_2(Y, Y - Y')$  is the  $K$ -theory of  $Y$  relative to  $Y - Y'$ . It follows from the existence of relative syntomic Chern classes, proved in [BdJ99], that the image of (3.5) is contained in the image of  $H^2_{ms}(Y, Y - Y', 2) \rightarrow H_{ms}(Y, 2)$  and it will therefore be enough to show that the relative modified syntomic cohomology group  $H^2_{ms}(Y, Y - Y', 2)$  is trivial. The modified syntomic cohomology can be written as the limit of cohomologies of certain cones [Bes99, 8.6.1]. Writing the associated long exact sequences, which are valid also for relative cohomology, we get

$$\begin{aligned} \cdots \rightarrow H^1_{rig}(Y_\kappa, Y_\kappa - Y'_\kappa/K) \rightarrow H^2_{ms}(Y, Y - Y', 2) \\ \rightarrow \text{Fil}^2 H^2_{dR}(Y_K, Y_K - Y'_K/K) \rightarrow \cdots \end{aligned}$$

The relative cohomologies on either side can be rewritten as cohomologies with support and since  $Y'$  is smooth over  $\mathcal{O}_K$  we can use purity (known for rigid cohomology by [Ber97, Corollaire 5.7]) to get

$$H^1_{rig}(Y_\kappa, Y_\kappa - Y'_\kappa/K) \cong H^1_{rig, Y'_\kappa}(Y_\kappa/K) = 0$$

and

$$\text{Fil}^2 H^2_{dR}(Y_K, Y_K - Y'_K/K) \cong \text{Fil}^2 H^2_{dR, Y'_K}(Y_K/K) \cong \text{Fil}^1 H^0_{dR}(Y'_K/K) = 0. \quad \blacksquare$$

COROLLARY 3.6: *With  $Y$  as in the previous lemma, if  $s, s' \in K_2(Y)$  have the same restriction to  $K_2(K(Y))$ , then  $ch_{2,2}(s) = ch_{2,2}(s')$ .*

*Proof of Theorem 3:* Let  $s \in K_2(Z)$ . We can write the restriction of  $s$  to the function field  $L(Z)$  as  $\sum \{f_i, g_i\}$  with  $f_i, g_i$  in  $L(Z)$ . Let  $\pi$  be a uniformizer of  $L$ .

We may write each  $h \in L(Z)$  as  $h_1 \pi^k$  where  $k \in \mathbb{Z}$  and the divisor of  $h_1$  does not include the special fiber  $Z_\kappa$ . Using that we may rewrite the restriction to  $L(Z)$  as

$$s|_{L(Z)} = \sum \{f_i, g_i\} + \{\pi, f^n\}, \quad \text{with } \text{ord}_{Z_\kappa} f_i = \text{ord}_{Z_\kappa} g_i = \text{ord}_{Z_\kappa} f = 0.$$

Furthermore, the tame symbols of  $s|_{L(Z)}$  must all be trivial. The tame symbol on  $Z_\kappa$  equals  $t_{Z_\kappa}(\pi, f^n) = f^n|_{Z_\kappa}$ , so  $f^n|_{Z_\kappa} = 1$ . It follows that we may write  $f^n - 1 = \pi^k h$  where  $k \in \mathbb{Z}_{>0}$ ,  $h \in L(Z)$  and  $\text{ord}_{Z_\kappa} h = 0$ . In  $K_2(L(Z))$  we therefore have

$$(3.6) \quad -k\{\pi, f^n\} = \{\pi^{-k}, f^n\} = \{\pi^{-k}(f^n - 1), f^n\} = \{h, f^n\}.$$

Let  $Y \subset Z$  be an affine open on which  $f_i, g_i, f$  and  $h$  become invertible, and let  $U$  be the corresponding basic wide open. It follows from Corollary 3.6 that

$$ch_{2,2}(s)|_Y = \sum ch_{2,2}(\{f_i, g_i\}) - \frac{1}{k}ch_{2,2}(\{h, f^n\}) \quad \text{in } H_{\text{ms}}^2(Y, 2).$$

Let  $[\eta]$  be the image of  $s \in K_2(X)$  in  $H_{\text{dR}}^1(Z_L)$ , represented by the one-form  $\eta$ . It follows from the considerations above that

$$(3.7) \quad [\eta]|_U = \sum_i [\eta(f_i, g_i)] - 1/k[\eta(h, f^n)] \in H^1(U).$$

We choose a branch of the  $p$ -adic logarithm such that  $\log(\pi) = 0$ . This implies that at any point  $x \in Z_L$  we have  $\log t_x(\pi, f^n) = 0$ . Since the tame symbols of  $s|_{L(Z)}$  are all 1 this implies that

$$(3.8) \quad \sum_i \log t_x(f_i, g_i) = 0, \quad \text{for all } x \in Z_L.$$

At this point we extend scalars to  $\mathbb{C}_p$  and set as usual  $C = Z_{\mathbb{C}_p}$ . Let  $\omega \in \Omega^1(C)$ . We need to show that

$$[\omega] \cup [\eta] = \sum_i \int_{(f_i)} \log g_i \cdot \omega.$$

The cup product formula of Serre tells us that the left hand side can be computed as  $\sum_{x \in C} \text{Res}_x(F_\omega \cdot \eta)$ . Here,  $F_\omega$  need only be a local integral of  $\omega$  around  $x$  and the result is independent of the choices. If we want to write the cup product as a sum of terms corresponding to the decomposition (3.7), however, these independent choices are not sufficient because the forms  $\eta(f_i, g_i)$  have residues along the annuli

$e$  so the result depends on the choice of local integrals. By choosing  $F_\omega$  to be the Coleman integral of  $\omega$ , however, this splitting,

$$(3.9) \quad [\omega] \cup [\eta] = \sum_i \sum_{e \in \text{End}(U)} \text{Res}_e F_\omega \cdot \eta(f_i, g_i) - \frac{1}{k} \sum_{e \in \text{End}(U)} \text{Res}_e F_\omega \cdot \eta(h, f^n),$$

becomes possible. This is because the Coleman integral is unique up to a constant and the sum of residues of a rigid form over all annuli ends is 0 by [Col89, Prop. 4.3]. We treat the last term first. According to Proposition 3.4 we have

$$\sum_{e \in \text{End}(U)} \text{Res}_e F_\omega \cdot \eta(h, f^n) = \sum_{x \in C} (\log t_x(h, f^n)) \cdot F_\omega(x) + \int_{(h)} \log f^n \cdot \omega.$$

While the left hand side is only defined because  $h$  and  $f^n$  are units in  $\mathcal{O}_Y$ , the right hand side is defined for any two rational functions and it factors through  $K_2$ . By (3.6) we see that the above expression equals

$$-k \left( \sum_{x \in C} (\log t_x(\pi, f^n)) \cdot F_\omega(x) + \int_{(\pi)} \log f^n \cdot \omega \right) = 0.$$

It now follows, again by Proposition 3.4, that the right hand side of (3.9) equals

$$\begin{aligned} \sum_i \left( \sum_{x \in C} (\log t_x(f_i, g_i)) \cdot F_\omega(x) + \int_{(f_i)} \log g_i \cdot \omega \right) \\ = \sum_i \int_{(f_i)} \log g_i \cdot \omega + \sum_{x \in C} F_\omega(x) \left( \sum_i \log t_x(f_i, g_i) \right). \end{aligned}$$

The last summand is 0 by (3.8) and so the theorem is proved. ■

#### 4. Residues and a reciprocity law

In this section we prove a reciprocity law, Corollary 4.11, which will be used in the next section to prove Proposition 3.4 and hence the main theorem. To state it, we will make a certain extension to the  $p$ -adic (or algebraic) notion of residue. We start by briefly recalling the setup, to be found in [Col89, II].

*Definition 4.1:* An open annulus is a rigid space over  $\mathbb{C}_p$  isomorphic to

$$A(r, s) := \{x \in \mathbb{C}_p : r < |x| < s\}$$

with  $r, s \in |\mathbb{C}_p^\times|$ . A uniformizing parameter for an annulus  $V$  is a rigid function giving an isomorphism of  $V$  with some  $A(r, s)$ .

Given an annulus  $V$  and a uniformizing parameter  $z$  on  $V$  we can define residues in the usual way. A rigid form  $\omega \in \Omega^1(V)$  can be written as  $\omega = \sum_{n=-\infty}^{\infty} a_n z^n dz$ . The residue of  $\omega$  with respect to  $z$  is the constant  $a_{-1}$ . Clearly the residue of  $dh$  is 0 if  $h \in A(V)$ . It can be shown that the residue is independent of the parameter  $z$  up to a (unique) sign. Clearly the sign is reversed if we switch from  $z$  to  $z^{-1}$ .

*Definition 4.2:* An orientation of an annulus is a choice of a residue function  $\text{Res}: \Omega^1(U) \rightarrow \mathbb{C}_p$ , equal to the residue function with respect to some uniformizing parameter. The reverse orientation is given by the function  $-\text{Res}$ . An annulus together with an orientation is called an oriented annulus.

The residue gives an isomorphism  $H^1(V) \rightarrow \mathbb{C}_p$ . Suppose  $f: V \rightarrow W$  is a rigid map between oriented annuli. Since  $f^*dA(W) \subset dA(V)$ , the following definition makes sense.

*Definition 4.3:* The degree of a map  $f$  is the unique number  $\deg f$  such that  $\text{Res}_V f^* \omega = \deg f \cdot \text{Res}_W \omega$ .

To extend the definition of residues to somewhat more general “functions” we make use of the following trivial linear algebra lemma.

**LEMMA 4.4:** *Let  $F$  be a field of characteristic different from 2. Let  $B$  be an  $F$  vector space and let  $r: B \rightarrow F$  be a non-zero linear map. Suppose we are given a bilinear pairing  $\langle \cdot, \cdot \rangle: \text{Ker}(r) \times B \rightarrow F$  whose restriction to  $\text{Ker}(r) \times \text{Ker}(r)$  is anti-symmetric. Then there is a unique anti-symmetric extension of  $\langle \cdot, \cdot \rangle$  to  $B \times B$ .*

*Proof:* Suppose we are given such an extension  $\ll \cdot, \cdot \gg$ . Choose  $x \in B$  such that  $r(x) \neq 0$ . For  $y, z \in B$  we have unique  $\alpha, \beta \in F$  and  $y', z' \in \text{Ker}(r)$  such that  $y = y' + \alpha x$  and  $z = z' + \beta x$ . Since  $\ll x, x \gg = 0$  by anti-symmetry we must have

$$\begin{aligned} \ll y, z \gg &= \ll y' + \alpha x, z' + \beta x \gg = \ll y', z' \gg + \beta \ll y', x \gg - \alpha \ll z', x \gg \\ (4.1) \qquad \qquad \qquad &= \langle y', z' \rangle + \beta \langle y', x \rangle - \alpha \langle z', x \rangle. \end{aligned}$$

This shows uniqueness. To show existence one only has to show that the formula (4.1) in fact defines an anti-symmetric bilinear extension of  $\langle \cdot, \cdot \rangle$ . Bilinearity is clear. To check anti-symmetry we substitute  $y' = z'$  and  $\alpha = \beta$  in (4.1) and easily get 0 using the anti-symmetry of  $\langle \cdot, \cdot \rangle|_{\text{Ker}(r) \times \text{Ker}(r)}$ . If  $y = y'$  and  $\alpha = 0$  we get in (4.1)

$$\ll y, z \gg = \langle y, z' \rangle + \beta \langle y, x \rangle = \langle y, z \rangle,$$



which shows that  $\langle\langle, \rangle\rangle$  indeed extends  $\langle, \rangle$ . ■

We apply this lemma to the following situation: Let  $V$  be an oriented annulus. We let  $B = \int \Omega^1(V)$ , the collection of all Coleman integrals of rigid forms on  $V$ . The map  $r$  will be given by  $r(F) := \text{Res}(dF)$ , which is well defined because  $dF \in \Omega^1(V)$ . Our pairing will be defined by  $\langle F, G \rangle := \text{Res } FdG$ . This is well defined when  $r(F) = 0$  as this is equivalent to  $F \in A(V)$ . If also  $G \in A(V)$ , then we have

$$0 = \text{Res } d(FG) = \text{Res}(FdG + GdF) = \langle F, G \rangle + \langle G, F \rangle,$$

showing the anti-symmetry. Our lemma therefore gives the following result.

PROPOSITION 4.5: *There is a unique anti-symmetric bilinear function,*

$$\text{ind}_V : \int \Omega^1(V) \times \int \Omega^1(V) \rightarrow \mathbb{C}_p,$$

such that  $\text{ind}_V(F, G) = \text{Res}_V(FdG)$  whenever  $F \in A(V)$ .

An immediate consequence of the uniqueness is the following functoriality property.

LEMMA 4.6: *If  $f: V \rightarrow W$  is a map of oriented annuli, then for any  $F, G \in \int \Omega^1(W)$  we have*

$$\text{ind}_V(f^*F, f^*G) = \text{deg } f \cdot \text{ind}_W(F, G).$$

*In particular, if  $W'$  is the annulus  $W$  with the reversed orientation, then we have  $\text{ind}_{W'}(F, G) = -\text{ind}_W(F, G)$ .*

We now globalize. Suppose  $U$  is a basic wide open in a curve  $C$ . Given two Coleman functions,  $F$  and  $G$ , such that their restrictions to all annuli ends  $e \in \text{End}(U)$  are of the type described above, we would like to describe the sum  $\sum_e \text{ind}_e(F, G)$ . The first result along these lines is

LEMMA 4.7: *If  $F \in A(U)$  and  $dG \in \Omega^1(U)$ , then  $\sum_{e \in \text{End}(U)} \text{ind}_e(F, G) = 0$ .*

*Proof:* As  $\text{Res}_e dF = 0$ , for all  $e \in \text{End}(U)$ , we see that  $\text{ind}_e(F, G) = \text{Res}_e(FdG)$  for all  $e$ . As  $FdG \in \Omega^1(U)$ , the result follows because by [Col89, Proposition 4.3], for any  $\omega \in \Omega^1(U)$ ,  $\sum_{e \in \text{End}(U)} \text{Res}_e \omega = 0$ . ■

PROPOSITION 4.8: *There is a canonical split exact sequence*

$$0 \rightarrow H^1(C) \xrightarrow{j^*} H^1(U) \xrightarrow{\text{Res}} \bigoplus_{e \in \text{End}(U)} \mathbb{C}_p.$$

*Proof:* The short exact sequence is just part of the standard long exact sequence in rigid cohomology

$$\begin{aligned} \cdots \rightarrow \bigoplus H_{\text{rig},x}^1(X_{\bar{F}_p}/\mathbb{C}_p) &\rightarrow H^1(X_{\bar{F}_p}/\mathbb{C}_p) \rightarrow H^1(Y_{\bar{F}_p}/\mathbb{C}_p) \\ &\rightarrow \bigoplus H_{\text{rig},x}^2(X_{\bar{F}_p}/\mathbb{C}_p) \rightarrow \cdots \end{aligned}$$

Here  $X$  and  $Y$  are the smooth  $\mathcal{O}_{\mathbb{C}_p}$  schemes that give rise to  $C$  and  $U$  as in section 2 and the direct sums are over all  $x \in X_{\bar{F}_p} - Y_{\bar{F}_p}$ . The first direct sum is 0 and in the second each summand is  $\mathbb{C}_p$  by purity. Explicitly, we can always represent a cohomology class in  $H^1(C)$  by a form of the second kind  $\omega$  with no poles on  $U$ . The map  $j^*$  is then given simply by restriction to  $U$ . The map  $\text{Res}$  is given by  $\eta \mapsto (\text{Res}_e \eta)_e$ . Considering the action of Frobenius we see that the eigenvalues on  $H^1(C)$  are, by the Weil conjectures for crystalline cohomology [KM74], algebraic integers with absolute value  $\sqrt{q}$ . On  $\bigoplus \mathbb{C}_p$  Frobenius acts as multiplication by  $q$ . This gives the splitting, which is easily seen to be independent of the choice of  $\phi$ . ■

We obtain a canonical projection  $\mathbf{p}: H^1(U) \rightarrow H^1(C)$ .

LEMMA 4.9: For any  $h \in A(U)$ ,  $\mathbf{p}(\text{dlog } h) = 0$ .

*Proof:* First we notice that if  $U' \hookrightarrow U$  is an injection of basic wide open spaces in  $C$ , then there is a commutative diagram

$$\begin{array}{ccc} H^1(U) & \longrightarrow & H^1(U') \\ \mathbf{p} \downarrow & & \mathbf{p} \downarrow \\ H^1(C) & \xlongequal{\quad} & H^1(C) \end{array}$$

We may consider an underlying affinoid  $Z$  in  $U$  and assume that  $\|h|_Z\| = 1$ . We then remove from  $U$  the residue discs of points where the reduction of  $h|_Z$  is 0 and obtain a new wide open  $U'$  with an underlying affinoid on which  $|h| = 1$ . The proof of [CdS88, Lemma 2.5.1] shows that for some Frobenius endomorphism  $\phi$  on  $U'$  of a large enough degree  $q$  we have  $(\phi^* - q) \text{dlog } h \in dA(U')$ . This gives what we want. ■

We can now state the main result of this section.

PROPOSITION 4.10: Let  $F$  and  $G$  be Coleman functions on a basic wide open  $U$  such that  $dF, dG \in \Omega^1(U)$ . Denote by  $[dF], [dG]$  the corresponding cohomology classes in  $H^1(U)$ . Then

$$\sum_{e \in \text{End}(U)} \text{ind}_e(F, G) = \mathbf{p}[dF] \cup \mathbf{p}[dG].$$

COROLLARY 4.11: *If  $h \in A(U)$  and  $F$  is a Coleman function on  $U$  with  $dF \in \Omega^1(U)$ , then*

$$\sum_{e \in \text{End}(U)} \text{ind}_e(F, \log h) = 0.$$

*Proof of Proposition 4.10:* It follows from Lemma 4.7 that setting

$$\langle [dF], [dG] \rangle := \sum_{e \in \text{End}(U)} \text{ind}_e(F, G)$$

gives a well defined pairing  $\langle \cdot, \cdot \rangle: H^1(U) \times H^1(U) \rightarrow \mathbb{C}_p$ . Let  $\phi: U \rightarrow U$  be a Frobenius endomorphism of degree  $q$ , preserving all ends. It is easy to see that  $\phi$  has degree  $q$  at all ends. It therefore follows from Lemma 4.6 that  $\langle \phi^*x, \phi^*y \rangle = q\langle x, y \rangle$  for any  $x, y \in H^1(U)$ . Let  $j^*$  be the map from Proposition 4.8. The formula of Serre for the cup product on  $H^1(C)$  implies that  $\langle j^*(x), j^*(y) \rangle = x \cup y$  if  $x, y \in H^1(C)$ . All we need therefore to show is that  $\langle H^1(U), \text{Ker } \mathbf{p} \rangle = 0$ . This is now clear from weight considerations: Suppose  $x \in \text{Ker } \mathbf{p}$ . Then  $\phi^*x = qx$ , which implies that for any  $y \in H^1(U)$  we have

$$\langle \phi^*y, x \rangle = q^{-1}\langle \phi^*y, \phi^*x \rangle = \langle y, x \rangle.$$

Thus, pairing with  $x$  provides a Frobenius invariant functional on  $H^1(U)$ . By Proposition 4.8 and its proof the eigenvalues of  $\phi$  on  $H^1(U)$  are either  $q$  or algebraic integers with absolute value  $\sqrt{q}$ . The functional above must therefore be 0. ■

*Remark 4.12:* Proposition 4.10 implies the main theorem of [Col89]. In particular, the projection  $\mathbf{p}$  extends the map in Theorem 4.10 there.

PROPOSITION 4.13: *Let  $G$  and  $H$  be Coleman functions on  $U$  such that  $G \in A(U)$ ,  $dH \in \Omega^1(U)$  and  $\text{Res}_e dH = 0$  at all  $e \in \text{End}(U)$ . Let  $I$  be a Coleman function such that  $dI = HdG$ . Let  $f \in A(U)$ . Then*

$$\sum_{e \in \text{End}(U)} \text{Res}_e GH \, d \log f = \sum_{e \in \text{End}(U)} \text{ind}_e(GH, \log f) = \sum_{e \in \text{End}(U)} \text{ind}_e(I, \log f).$$

*Proof:* The assumptions on  $H$  imply that  $H|_e \in A(e)$  for all  $e$ , and therefore the same holds true for  $GH$ . This implies the first equality. It also shows that  $dI|_e \in \Omega^1(e)$  and therefore the expression  $\text{ind}_e(I, \log f)$  makes sense. We have  $GdH \in \Omega^1(U)$  and we can find a Coleman function  $J$  with  $dJ = GdH$ . As  $d(GH) = GdH + HdG = dJ + dI$  we see that  $GH = I + J + \text{Const}$  (this is just a complicated way to write integration by parts). Corollary 4.11 shows that  $\sum_e \text{ind}_e(J + \text{Const}, \log f) = 0$ , giving the result. ■

### 5. Computation of the regulator

To prove Proposition 3.4, and hence the main theorem, we will introduce a new regulator, which we will show equals each side of (3.4) in turn.

PROPOSITION 5.1: *Let  $U$  be a basic wide open,  $f, g \in A(U)$  and  $\omega \in \Omega^1(U)$  with  $\text{Res}_e \omega = 0$  for all ends  $e$  of  $U$ . Let  $F_\omega$  be a Coleman integral of  $\omega$  and choose a Coleman integral  $\int F_\omega \text{dlog } g$ . Then the expression*

$$\rho(f, g)([\omega]) := \sum_{e \in \text{End}(U)} \text{ind}_e(\log f, \int F_\omega \text{dlog } g)$$

is well defined and depends only on the cohomology class  $[\omega] \in H^1(C) \subset H^1(U)$ . It therefore defines a map  $\rho: A(U) \otimes A(U) \rightarrow \text{Hom}(H^1(C), \mathbb{C}_p)$ .

*Proof:* The condition on  $\omega$  implies that  $F_\omega|_e \in A(e)$  for all  $e$ , showing that the expression  $\text{ind}_e(\log f, \int F_\omega \text{dlog } g)$  is well defined. If  $\omega$  is exact, then  $F_\omega \in A(U)$  hence  $F_\omega \text{dlog } g \in \Omega^1(U)$  and  $\sum_e \text{ind}_e(\log f, \int F_\omega \text{dlog } g) = 0$  by Corollary 4.11.

■

Remark 5.2: Because, morally,

$$\rho(f, g)([\omega]) = \sum_i \text{Res}_{e_i} \log f \text{dlog } g F_\omega = [\log f \text{dlog } g] \cup [\omega],$$

this regulator should be compared with the complex regulator (1.1).

PROPOSITION 5.2: *Let  $f, g$  and  $\omega$  be as above and let  $\eta(f, g)$  be as in Proposition 3.3. Then*

$$\rho(f, g)([\omega]) = \sum_{e \in \text{End}(U)} \text{Res}_e F_\omega \eta(f, g).$$

Before giving the actual proof we give a heuristic proof. This proof demonstrates that the proposition would be very easy if we could treat logs as good functions, and in particular take their residues. The actual proof repeats the same considerations using the local indices.

The idea is that by the trick of Remark 3.1, and in particular (3.1), we have

$$\begin{aligned} \left(1 - \frac{\phi^*}{q^2}\right) (\log f \text{dlog } g) &= \left(1 - \frac{\phi^*}{q}\right) \log f \cdot \frac{\phi^*}{q} \text{dlog } g + \log f \cdot \left(1 - \frac{\phi^*}{q}\right) \text{dlog } g \\ &= \frac{1}{q^2} \log f_0 \text{dlog } \phi^* g + \frac{1}{q} \log f \text{dlog } g_0 \\ &= \left(1 - \frac{\phi^*}{q^2}\right) \eta(f, g) + \frac{1}{q} d(\log f \log g_0) + d(\ ), \end{aligned}$$

where the last equality follows from Proposition 3.3. In other words, the two forms  $(1 - \phi^*/q^2)(\log f \, d\log g)$  and  $(1 - \phi^*/q^2)\eta$  “have the same cohomology class”. Now one should argue that  $(1 - \phi^*/q^2)$  is invertible on cohomology. Let us make this a bit more reasonable. We have

$$\phi^*\omega \cup \left(1 - \frac{\phi^*}{q^2}\right)\eta = \phi^*\omega \cup \eta - \frac{1}{q}\omega \cup \eta = \left(\phi^* - \frac{1}{q}\right)\omega \cup \eta.$$

As the same is valid with  $\log f \, d\log g$  in place of  $\eta$  and as  $(\phi^* - 1/q)$  is invertible on  $H^1(C)$  we are done. Now for the actual proof.

*Proof of Proposition 5.3:* We note first that both sides of the equation depend only on the cohomology class of  $\omega$ . We will evaluate both sides on  $(\phi^* - 1/q)\omega$ . Put  $\langle \epsilon, \theta \rangle = \sum_e \text{Res}_e F_\epsilon \theta$  for shorthand, and also  $\eta = \eta(f, g)$ ,  $\eta_0 = \eta_0(f, g)$ . As  $(1 - \phi^*/q^2)\eta = \eta_0 + d(\cdot)$  by (3.3) we have

$$(5.1) \quad \langle \eta, (\phi^* - 1/q)\omega \rangle = \langle \eta, \phi^*\omega \rangle - (1/q^2)\langle \phi^*\eta, \phi^*\omega \rangle = \langle \eta_0, \phi^*\omega \rangle.$$

Let  $\theta = \phi^*\omega$ . Then  $\theta$  is a one-form on  $U$  whose residues along the ends of  $U$  is 0. We have

$$\begin{aligned} \langle \eta_0, \theta \rangle &= \frac{1}{q^2} \sum_e \text{Res}_e \log f_0 F_\theta \, d\log \phi^*g - \frac{1}{q} \sum_e \text{Res}_e F_\theta \log g_0 \, d\log f \\ &= \frac{1}{q^2} \sum_e \text{ind}_e(\log f_0, \int F_\theta \, d\log \phi^*g) + \frac{1}{q} \sum_e \text{ind}_e(\log f, \int F_\theta \, d\log g_0), \end{aligned}$$

where we have used Proposition 4.13 to rewrite the second summand. Making sure that the local indices make sense at each stage, we now have

$$\begin{aligned} \langle \eta_0, \theta \rangle &= \frac{1}{q} \sum_e \text{ind}_e \left( \log f - \frac{\phi^*}{q} \log f, \int F_\theta \, d\log \phi^*g \right) \\ &\quad + \sum_e \text{ind}_e \left( \log f, \int F_\theta \left( d\log g - \frac{\phi^*}{q} d\log g \right) \right) \\ &= \sum_e \text{ind}_e \left( \log f, \int F_\theta \frac{\phi^*}{q} d\log g \right) - \sum_e \text{ind}_e \left( \frac{\phi^*}{q} \log f, \int F_\theta \frac{\phi^*}{q} d\log g \right) \\ &\quad + \sum_e \text{ind}_e \left( \log f, \int F_\theta \, d\log g \right) - \sum_e \text{ind}_e \left( \log f, \int F_\theta \frac{\phi^*}{q} d\log g \right). \end{aligned}$$

The first and last terms cancel out, leaving us with

$$= \sum_e \text{ind}_e \left( \log f, \int F_\theta \, d\log g \right) - \sum_e \text{ind}_e \left( \frac{\phi^*}{q} \log f, \int F_\theta \frac{\phi^*}{q} d\log g \right),$$

and using the fact that  $\theta = \phi^*\omega$  and the equivariance of the local index proved in Lemma 4.6

$$\begin{aligned} &= \sum_e \text{ind}_e \left( \log f, \int F_{\phi^*\omega} \, d\log g \right) - \frac{1}{q} \sum_e \text{ind}_e \left( \log f, \int F_\omega \, d\log g \right) \\ &= \sum_e \text{ind}_e \left( \log f, \int F_{(\phi^* - 1/q)\omega} \, d\log g \right) = \rho(f, g)((\phi^* - 1/q)[\omega]). \end{aligned}$$

Together with (5.1) we get the equation

$$\rho(f, g)((\phi^* - 1/q)[\omega]) = \sum_e \text{Res}_e \eta(f, g) F_{(\phi^* - 1/q)\omega},$$

proving the proposition for  $(\phi^* - 1/q)\omega$  instead of  $\omega$ , and therefore for any form in the cohomology class  $[(\phi^* - 1/q)\omega]$ . But the operator  $(\phi^* - 1/q)$  is invertible on  $H^1(C)$  so the proposition follows. ■

The last step is to compute the local indices on an annulus  $e$  occurring in the definition of  $\rho(f, g)([\omega])$  in terms of the singular points of  $f$  and  $g$  inside the corresponding disc. By scaling it is enough to consider the open unit disc  $D = \{|z| < 1\}$ . We need to recall the following.

LEMMA 5.4: *Let  $e = A(s, 1) \subset D$  be an open annulus.*

1. *Let  $\omega$  be a rigid differential form on  $D$  which is analytic on  $e$  and has at most a finite number of poles otherwise. Then  $\text{Res}_e \omega = \sum_{x \in D} \text{Res}_x \omega$ .*
2. *Let  $f \in A(D)^\times$ . Then  $\log(f) \in A(D)$ .*

*Proof:* The first part is just [Col89, Prop. 2.3]. The disc  $D$  is in particular a basic wide open space and  $e$  is its unique annulus end. By the first part we have  $\text{ord}_e(f) := \text{Res}_e \, d\log f = 0$ . It therefore follows from [Col89, Lemma 4.8] that  $|cf - 1|$  is bounded above by  $\alpha < 1$  for some  $c \in \mathbb{C}_p$ . It follows that  $\log(cf) = \log c + \log f$  is a convergent power series in  $cf - 1$  and is therefore analytic. ■

PROPOSITION 5.5: *Let  $f$  and  $g$  be meromorphic functions on  $D$  with a finite number of poles and zeros. Let  $\omega \in \Omega^1(D)$  and choose an integral  $F_\omega$ . Let  $e = A(s, 1)$  be an annulus on which both  $f$  and  $g$  are invertible. Let  $H$  be a Coleman function on  $D$  such that  $dH = \log g \cdot \omega$  and let  $G = \log g \cdot F_\omega - H$ , so that  $dG = F_\omega \, d\log g$ . Then*

$$\text{ind}_e(\log f, G) = \sum_{x \in D} (\log t_x(f, g) \cdot F_\omega(x) + \text{ord}_x(f) \cdot H(x)),$$

where we use Definition 2.7 to compute  $H(x)$  when needed.

*Proof:* First we claim that the truth of the proposition is independent of the choice of  $H$ . Indeed, if we replace  $H$  by  $H + C$ , where  $C$  is a constant, then  $G$  is replaced by  $G - C$ . With the convention of Definition 2.7 this is true even at the support of the divisor of  $g$ . Thus, the claim follows from

$$\text{ind}_e(\log f, C) = -\text{Res}_e C \, d\log f = -C \sum_{x \in D} \text{ord}_x(f),$$

where the last equality follows from Lemma 5.4. Let

$$S = \{x \in D: \text{ord}_x(f) \neq 0 \text{ or } \text{ord}_x(g) \neq 0\}.$$

The proof will be by induction on the size of  $S$ . When it is 0 both sides vanish, the right hand side trivially and the left hand side because by Lemma 5.4 both  $\log f$  and  $G$  are analytic on  $D$ . If  $S \neq \emptyset$ , then we may assume that  $0 \in S$  by translation. When  $S = \{0\}$  the proof will be given below. Suppose then that  $S$  has at least 2 elements. Let  $r = \max_{x \in S} |x|$  and set  $S_r = \{x \in S, |x| = r\}$ . There is some  $t < r$  such that

$$W := \{z \in D: \forall x \in S, |z - x| > t\}$$

is a basic wide open disjoint from  $S$ . It is obtained by removing from  $\mathbb{P}^1$  closed discs  $D_i, i = 0, \dots, k$ , with  $D_0 = \{|z| \geq 1\}$ ,  $D_i = D[a_i, t]$  for some  $a_i \in D$  for  $i \geq 1$ , and such that  $S \subset \bigcup_{i=1}^k D_i$ . Let  $e_i, i = 0, \dots, k$  be the corresponding annuli ends of  $W$ . Then  $e_0$  is just  $e$  with the reversed orientation. Therefore, Corollary 4.11 and Lemma 4.6 imply that

$$\text{ind}_e(\log f, G) = \sum_{i=1}^k \text{ind}_{e_i}(\log f, G).$$

This shows that the statement of the proposition is true for  $D$  if it is true for the discs  $D(a_i, t)$  after translation and scaling to the open unit disc. Since 0 and any  $x \in S_r$  are not in the same disc  $D_i$ , we are done by induction.

It remains to prove the result when  $S = \{0\}$ . Both sides of the equation are bilinear in  $f$  and  $g$  (with respect to multiplication) and linear in  $F_\omega$ . This follows from our claim at the beginning of the proof and from Lemma 2.8. It is therefore sufficient to consider the cases listed below.

If  $\text{ord}_0 f \neq 0$  and  $\text{ord}_0 g = 0$ , then  $dG = F_\omega \text{dlog } g \in \Omega^1(D)$ , hence  $G \in A(D)$ . It follows that

$$\begin{aligned} \text{ind}_e(\log f, G) &= -\text{Res}_e G \text{dlog } f = -\text{ord}_0(f) \cdot G(0) \\ &= \text{ord}_0(f) \cdot (H(0) - \log g(0) \cdot F_\omega(0)) \\ &= \log t_0(f, g) \cdot F_\omega(0) + \text{ord}_0(f) \cdot H(0). \end{aligned}$$

If  $\text{ord}_0 f = 0$  and  $\text{ord}_0 g \neq 0$ , then  $\log f \in A(D)$ , so

$$\begin{aligned} \text{ind}_e(\log f, G) &= \text{Res}_e \log f dG = \text{Res}_e \log f F_\omega \text{dlog } g = \text{Res}_0 \log f F_\omega \text{dlog } g \\ &= \text{ord}_0(g) \cdot \log f(0) \cdot F_\omega(0) = \log t_0(f, g) \cdot F_\omega(0). \end{aligned}$$

If  $f(z) = g(z) = z$  and  $F_\omega(0) = 0$ , then we still have  $dG \in \Omega^1(D)$  so we can compute as in the first case. Our convention exactly says that because  $F_\omega(0) = 0$  we have  $-G(0) = H(0)$ . It follows that

$$\text{ind}_e(\log f, G) = H(0) = \log t_0(z, z) \cdot F_\omega(0) + \text{ord}_0(z) \cdot H(0).$$

Finally, if  $f(z) = g(z) = z$  and  $F_\omega = C$ , where  $C$  is a constant function, then  $dH = 0$  so  $H$  is constant and  $G = C \log z - H$ . It follows that

$$\begin{aligned} \text{ind}_e(\log f, G) &= \text{ind}_e(\log z, C \log z - H) \\ &= -\text{ind}_e(\log z, H) = \text{Res}_e H \text{dlog } z = H = H(0). \end{aligned}$$

The proof is complete. ■

*Proof of Proposition 3.4:* Let  $D_e$  be the residue disc corresponding to the annulus  $e$ . Then

$$\begin{aligned} &\sum_{e \in \text{End}(U)} \text{Res}_e(F_\omega \cdot \eta(f, g)) \\ &= \sum_{e \in \text{End}(U)} \text{ind}_e(\log f, \int F_\omega \text{dlog } g) \quad \text{Prop. 5.3} \\ &= \sum_{e \in \text{End}(U)} \sum_{x \in D_e} \left( (\log t_x(f, g)) \cdot F_\omega(x) + \text{ord}_x(f) \cdot \left( \int \log g \cdot \omega \right)(x) \right) \quad \text{Prop. 5.5} \\ &= \sum_{x \in C} (\log t_x(f, g)) \cdot F_\omega(x) + \int_{(f)} \log g \cdot \omega. \quad \blacksquare \end{aligned}$$



**References**

- [BdJ99] A. Besser and R. de Jeu, *The syntomic regulator for  $K$ -theory of fields*, Work in progress, 1999.
- [Bei80] A. A. Beilinson, *Higher regulators and values of  $L$ -functions of curves*, *Functional Analysis and its Applications* **14** (1980), 116–118.
- [Ber96] P. Berthelot, *Cohomologie rigide et cohomologie rigide a support propres, premier partie*, Preprint 96-03 of the University of Rennes, available at <http://www.maths.univ-rennes1.fr/~berthelo/>, 1996.
- [Ber97] P. Berthelot, *Finitude et pureté cohomologique en cohomologie rigide*, *Inventiones Mathematicae* **128** (1997), 329–377, with an appendix in English by A. J. de Jong.
- [Bes99] A. Besser, *Syntomic regulators and  $p$ -adic integration I: rigid syntomic regulators*, this volume.
- [CdS88] R. Coleman and E. de Shalit,  *$p$ -adic regulators on curves and special values of  $p$ -adic  $L$ -functions*, *Inventiones Mathematicae* **93** (1988), 239–266.
- [Col89] R. Coleman, *Reciprocity laws on curves*, *Compositio Mathematica* **72** (1989), 205–235.
- [Gro90] M. Gros, *Régulateurs syntomiques et valeurs de fonctions  $L$   $p$ -adiques. I*, *Inventiones Mathematicae* **99** (1990), 293–320, with an appendix by Masato Kurihara..
- [Har77] G. Harder, *Die Kohomologie  $S$ -arithmetischer Gruppen über Funktionenkörpern*, *Inventiones Mathematicae* **42** (1977), 135–175.
- [KM74] N. Katz and W. Messing, *Some consequences of the Riemann hypothesis for varieties over finite fields*, *Inventiones Mathematicae* **23** (1974), 73–77.
- [Niz97] W. Nizioł, *On the image of  $p$ -adic regulators*, *Inventiones Mathematicae* **127** (1997), 375–400.